A Minimal Poset Resolution of Stable Ideals

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Abstract

We use the theory of poset resolutions to give an alternate construction for the minimal free resolution of an arbitrary stable monomial ideal in the polynomial ring whose coefficients are from a field. This resolution is recovered by utilizing a poset of Eliahou-Kervaire admissible symbols associated to a stable ideal. The structure of the poset under consideration is quite rich and in related analysis, we exhibit a regular CW complex which supports a minimal cellular resolution of a stable monomial ideal.

1 Introduction

Let $R = \Bbbk[x_1, \ldots, x_d]$, where $\Bbbk$ is a field and R is considered with its standard $\mathbb{Z}^d$ grading (multigrading). A monomial ideal $N \subseteq R$ is called *stable* if for every monomial $m \in N$, the monomial $m \cdot x_i/x_r \in N$ for each $1 \leq i < r$, where $r = \max\{k : x_k \text{ divides } m\}$. The class of stable ideals, introduced by Eliahou and Kervaire in [14] is arguably the most well-studied class of monomial ideals. In their work, Eliahou and Kervaire construct the minimal free resolution of an arbitrary stable monomial ideal by identifying basis elements of the free modules present in the resolution – which they call admissible symbols – and by describing how the maps within the resolution act on these admissible symbols.

Among others, the class of stable ideals contains the so-called Borel-fixed [12] ideals as a subclass, which are fundamental to Gröbner Basis Theory. Certain homological properties of stable ideals are analyzed by Herzog and Hibi [15], where it is shown that stable ideals are componentwise linear. More

recently, distinct topological methods have been used to describe the minimal free resolution of an arbitrary stable ideal [1, 16].

We construct the minimal free resolution of an arbitrary stable ideal N using the theory of *poset resolutions*, introduced by the author in [9]. Specifically, we define a poset $(P_N, <)$ on the admissible symbols of Eliahou and Kervaire by taking advantage of a decomposition property unique to the monomials contained in stable ideals. In our first main result, Theorem 2.5, we recover the Eliahou-Kervaire minimal free resolution of a stable ideal as a poset resolution. The value of this technique is that the maps in the resolution are shown to act on the basis elements of the free modules in a way that mirrors the covering relations present in P_N. Taking the lattice-linear ideals of [9] into consideration, poset resolutions therefore provide a common perspective from which to view the minimal resolutions of three large and well-studied classes of monomial ideals; stable ideals, Scarf ideals [2] and ideals having a linear resolution [13, 15].

An additional advantage of the method described herein is that for a fixed stable ideal, combinatorial information from the poset of admissible symbols is transferred to topological information of a CW complex. Specifically, the poset P_N is a *CW poset* in the sense of Björner [5], which allows its interpretation as the face poset of a regular CW complex X_N. In our second main result, Theorem 6.4, we show that X_N supports a minimal cellular resolution of the stable ideal N. By utilizing this combinatorial connection, we are able to provide a minimal cellular resolution of N in a considerably more explicit manner than in three previous methods; the construction of Batzies and Welker [1] which uses the tools of Discrete Morse Theory, the result due to Mermin [16] that the Eliahou-Kervaire resolution is cellular and the technique of Sinefakopoulos [19] which produces a polyhedral cell complex that supports the minimal free resolution of (the more restricted class of) Borel-fixed ideals generated in one degree. In addition, Corso and Nagel in [11], construct cellular resolutions of strongly stable ideals generated in degree two and separately, Nagel and Reiner in [18], construct cellular resolutions of strongly stable ideals generated in one degree. To the author's knowledge, connections between these methods and the one contained in this paper have not been formally studied.

In Section 2, we review the fundamentals of poset resolutions and define the poset of admissible symbols P_N. Section 3 provides the details of the (dual) shellability of P_N. The intrinsic CW poset structure and properties of the sequence of vector spaces and maps which is produced by the methods of

[9] are exhibited in Section 4. The proof of Theorem 2.5 is given in Section 5 and Section 6 contains the description of the regular CW complex X_N and the proof of Theorem 6.4.

Throughout the paper, we assume that the topolological and combinatorial notions of posets, order complexes, CW complexes and face posets are familiar to the reader.

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2 Poset Resolutions and Stable Ideals

Let $(P, <)$ be a finite poset with set of atoms A and write $\beta \lessdot \alpha$ if $\beta < \alpha$ and there is no $\gamma \in P$ such that $\beta < \gamma < \alpha$. We say that β is *covered by* α in this situation. For $\alpha \in P$, write the order complex of the associated open interval as $\Delta_\alpha = \Delta(\hat{0}, \alpha)$. In [9], the collection of simplicial complexes

$$\{\Delta_\alpha : \alpha \in P\}$$

is used to construct a sequence of vector spaces and vector space maps

$$\mathcal{D}(P) : \cdots \longrightarrow \mathcal{D}_i \xrightarrow{\varphi_i} \mathcal{D}_{i-1} \longrightarrow \cdots \longrightarrow \mathcal{D}_1 \xrightarrow{\varphi_1} \mathcal{D}_0.$$

In this construction, $\mathcal{D}_0 = \widetilde{H}_{-1}(\{\varnothing\}, \Bbbk)$, a one-dimensional vector space.

For $i \geq 1$, the vector space $\mathcal{D}_i$ is defined as

$$\mathcal{D}_i = \bigoplus_{\alpha \in P \smallsetminus \{\hat{0}\}} \mathcal{D}_{i,\alpha},$$

where $\mathcal{D}_{i,\alpha} = \widetilde{H}_{i-2}(\Delta_\alpha, \Bbbk)$. In particular, the vector space $\mathcal{D}_1$ has its nontrivial components indexed by the set of atoms A, and the map $\varphi_1 : \mathcal{D}_1 \to \mathcal{D}_0$ is defined componentwise as $\varphi_1|_{\mathcal{D}_{1,\alpha}} = \mathrm{id}_{\widetilde{H}_{-1}(\{\varnothing\},\Bbbk)}$.

For $i \geq 2$, the maps φ_i are defined using the Mayer-Vietoris sequence in reduced homology for the triple

$$\left(\mathbf{D}_\lambda, \quad \bigcup_{\substack{\beta \lessdot \alpha \\ \lambda \neq \beta}} \mathbf{D}_\beta, \quad \Delta_\alpha \right) \tag{2.1}$$

where $\mathbf{D}_\lambda = \Delta(\hat{0}, \lambda]$ for all $\lambda \lessdot \alpha$. For notational simplicity, when $\lambda \lessdot \alpha$ let

$$\Delta_{\alpha,\lambda} = \mathbf{D}_\lambda \cap \left(\bigcup_{\substack{\beta \lessdot \alpha \\ \lambda \neq \beta}} \mathbf{D}_\beta \right)$$

so that $\iota : \widetilde{H}_{i-3}(\Delta_{\alpha,\lambda}, \Bbbk) \to \widetilde{H}_{i-3}(\Delta_\lambda, \Bbbk)$ is the map induced in homology by the inclusion map and

$$\delta^{\alpha,\lambda}_{i-2} : \widetilde{H}_{i-2}(\Delta_\alpha, \Bbbk) \to \widetilde{H}_{i-3}(\Delta_{\alpha,\lambda}, \Bbbk)$$

is the connecting homomorphism from the Mayer-Vietoris sequence in homology of (2.1). For $i \geq 2$ the map $\varphi_i : \mathcal{D}_i \to \mathcal{D}_{i-1}$ is defined componentwise by

$$\varphi_i|_{\mathcal{D}_{i,\alpha}} = \sum_{\lambda \lessdot \alpha} \varphi_i^{\alpha,\lambda}$$

where

$$\varphi_i^{\alpha,\lambda} : \mathcal{D}_{i,\alpha} \to \mathcal{D}_{i-1,\lambda}$$

is the composition $\varphi_i^{\alpha,\lambda} = \iota \circ \delta^{\alpha,\lambda}_{i-2}$.

We now describe the process by which the sequence of vector spaces $\mathcal{D}(P)$ is transformed into a sequence of multigraded modules. For a monomial $m = x_1^{\mathbf{a}_1} \cdots x_d^{\mathbf{a}_d} \in R$ we write $\mathrm{mdeg}(m) = (\mathbf{a}_1, \ldots, \mathbf{a}_d)$ and $\deg_{x_\ell}(m) = \mathbf{a}_\ell$ for $1 \leq \ell \leq d$. Assuming the existence of a map of partially ordered sets $\eta : P \longrightarrow \mathbb{N}^n$, the sequence of vector spaces $\mathcal{D}(P)$ is *homogenized* to produce

$$\mathcal{F}(\eta) : \cdots \longrightarrow \mathcal{F}_t \xrightarrow{\partial_t} \mathcal{F}_{t-1} \longrightarrow \cdots \longrightarrow \mathcal{F}_1 \xrightarrow{\partial_1} \mathcal{F}_0,$$

a sequence of free multigraded R-modules and multigraded R-module homomorphisms. Indeed, homogenization of $\mathcal{D}(P)$ is carried out by constructing $F_0 = R \otimes_\Bbbk \mathcal{D}_0$ and multigrading the result with $\mathrm{mdeg}(x^{\mathbf{a}} \otimes v) = \mathbf{a}$ for each $v \in \mathcal{D}_0$. Similarly, for $i \geq 1$, we set

$$\mathcal{F}_i = \bigoplus_{\hat{0} \neq \lambda \in P} \mathcal{F}_{i,\lambda} = \bigoplus_{\hat{0} \neq \lambda \in P} R \otimes_\Bbbk \mathcal{D}_{i,\lambda}$$

where the grading is defined as $\mathrm{mdeg}(x^{\mathbf{a}} \otimes v) = \mathbf{a} + \eta(\lambda)$ for each $v \in \mathcal{D}_{i,\lambda}$. The differential in this sequence of multigraded modules is defined componentwise in homological degree 1 as

$$\partial_1|_{F_{1,\lambda}} = x^{\eta(\lambda)} \otimes \varphi_1|_{\mathcal{D}_{1,\lambda}}$$

and for $i \geq 1$, the map $\partial_i : \mathcal{F}_i \longrightarrow \mathcal{F}_{i-1}$ is defined as

$$\partial_i|_{\mathcal{F}_{i,\alpha}} = \sum_{\lambda \lessdot \alpha} \partial_i^{\alpha,\lambda}$$

where $\partial_i^{\alpha,\lambda} : \mathcal{F}_{i,\alpha} \longrightarrow \mathcal{F}_{i-1,\lambda}$ takes the form $\partial_i^{\alpha,\lambda} = x^{\eta(\alpha)-\eta(\lambda)} \otimes \varphi_i^{\alpha,\lambda}$ for $\lambda \lessdot \alpha$. The sequence $\mathcal{F}(\eta)$ approximates a free resolution of the multigraded module R/M where M is the ideal in R generated by the monomials

$$\{x^{\eta(a)} : a \in A\}$$

whose multidegrees are given by the images of the atoms of P.

Definition 2.2. *[9] If $\mathcal{F}(\eta)$ is an acyclic complex of multigraded modules, then we say that it is a* poset resolution *of the ideal M.*

Throughout the remainder of the paper, N will denote a stable monomial ideal in R and we write $G(N)$ as the unique minimal generating set of N. For a monomial $m \in R$ set

$$\max(m) = \max\{k \mid x_k \text{ divides } m\}$$

and

$$\min(m) = \min\{k \mid x_k \text{ divides } m\}.$$

To describe further the class of stable ideals, let $[d-1] = \{1, \ldots, d-1\}$, for $I \subseteq [d-1]$ let $\max(I) = \max\{i \mid i \in I\}$ and write $x_I = \prod_{i \in I} x_i$.

In Lemma 1.2 of [14], Eliahou and Kervaire prove that a monomial ideal N is stable if and only if for each monomial $m \in N$ there exists a unique $n \in G(N)$ with the property that $m = n \cdot y$ and $\max(n) \leq \min(y)$. We adopt the language and notation introduced in the paper of Eliahou and Kervaire, and refer to n as the *unique decomposition* of the monomial m. Following their convention, we encode this property in a *decomposition map* $\mathbf{g} : M(N) \longrightarrow G(N)$ where $M(N)$ is the collection of monomials of N and $\mathbf{g}(m) = n$.

Definition 2.3. [14] *An* admissible symbol *is an ordered pair (I, m) which satisfies* $\max(I) < \max(m)$*, where $m \in G(N)$ and $I \subseteq [d-1]$.*

Definition 2.4. *The* poset of admissible symbols *is the set P_N of all admissible symbols associated to N, along with the symbol $\hat{0} = (\varnothing, 1)$ which is defined to be the minimum element of P_N. The partial ordering on P_N is*

$$(J,n) \leq (I,m) \iff \begin{array}{l} J \subseteq I \text{ and there exists} \\ C \subseteq I \setminus J \text{ so that } n = \mathbf{g}(x_C m) \end{array}$$

when both symbols are admissible.

In the case when $(J,n) < (I,m)$ and $I = J \cup \{\ell\}$ for some ℓ, then we write $(J,n) \lessdot (I,m)$ to describe the covering that occurs in P_N. As constructed, we have $\hat{0} \lessdot (\varnothing, m)$ for every $m \in G(N)$. We are now in a position to state our first main result.

Theorem 2.5. *Suppose that N is a stable monomial ideal with poset of admissible symbols P_N and define the map $\eta : P_N \longrightarrow \mathbb{N}^n$ so that $(I,m) \mapsto$ $\mathrm{mdeg}(x_I m)$. Then the complex $\mathcal{F}(\eta)$ is a minimal poset resolution of R/N.*

In order to prove Theorem 2.5, we first describe the combinatorial structure of P_N and then exhibit the connection between the complex $\mathcal{F}(\eta)$ and the minimal free resolution of the stable ideal N constructed by Eliahou and Kervaire in [14].

3 The Shellability of P_N

We begin this section by recalling some general facts regarding the shellability of partially ordered sets. Recall that a poset P is called *shellable* if the facets of its order complex $\Delta(P)$ can be arranged in a linear order $F_1, F_2, \ldots, F_t$ in such a way that the subcomplex

$$\left(\bigcup_{i=1}^{k-1} F_i \right) \bigcap F_k$$

is a nonempty union of maximal proper faces of F_k for $k = 2, \ldots, t$. Such an ordering of facets is called a *shelling*.

Definition 3.1. *Let $\mathcal{E}(P)$ denote the collection of edges in the Hasse diagram of a poset P. An* edge labeling *of P is a map $\lambda : \mathcal{E}(P) \longrightarrow \Lambda$ where Λ is some poset.*

For $\sigma = a_1 \lessdot \cdots \lessdot a_k$, a maximal chain of P, the *edge label* of σ is the sequence of labels $\lambda(\sigma) = (\lambda(a_1 \lessdot a_2), \ldots, \lambda(a_{k-1} \lessdot a_k))$.

Definition 3.2. *An edge labeling λ is called an* EL-labeling (edge lexicographical labeling) *if for every interval $[x, y]$ in P,*

1. *There is a unique maximal chain σ in $[x, y]$, such that the labels of σ form an increasing sequence in Λ. We call σ the unique increasing maximal chain in $[x, y]$.*
2. *$\lambda(\sigma) < \lambda(\sigma')$ under the lexicographic partial ordering in Λ for all other maximal chains σ' in $[x, y]$.*

A graded poset that admits an EL-labeling is said to be EL-shellable (edge lexicographically shellable).

We further recall the following fundamental result of Björner and Wachs.

Theorem 3.3. *[7] EL-shellable posets are shellable.*

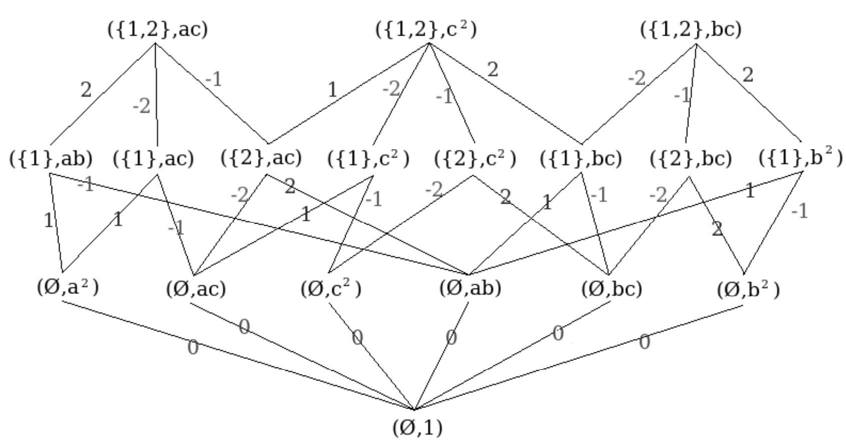
We now define an edge labeling of the poset of admissible symbols P_N.

Definition 3.4. *Let $\lambda : P_N \to \mathbb{Z}$ take the form*

$$\lambda\left((J, n) \lessdot (I, m)\right) = \begin{cases} 0 & \text{if } n = 1 \\ -\ell & \text{if } n = m \\ \ell & \text{if } n \neq m, \end{cases}$$

where $\{\ell\} = I \smallsetminus J$.

Example 3.5. *The labeled Hasse diagram for the poset of admissible symbols, P_N, of the stable ideal $N = \langle a, b, c\rangle^2 = \langle a^2, ab, ac, b^2, bc, c^2\rangle$ is*



Recall that given a poset P, the *dual poset* P^* has an underlying set identical to that of P, with $x < y$ in P^* if and only if $y < x$ in P. Further, an edge labeling of a poset P may also be viewed as an edge labeling of its dual poset and P is said to be *dual shellable* if P^* is a shellable poset.

Theorem 3.6. *The poset P_N is dual EL-shellable with λ defined as above.*

Before turning to the proof of Theorem 3.6, we discuss some properties of the decomposition map $\mathbf{g}$ and the edge labeling λ.

Remarks 3.7.

1. *[14, Lemma 1.3] For any monomial w and any monomial $m \in N$, we have $\mathbf{g}(w\mathbf{g}(m)) = \mathbf{g}(wm)$ and $\max(\mathbf{g}(wm)) \leq \max(\mathbf{g}(m))$. We refer to the first property as the* associativity *of* $\mathbf{g}$.

2. *Suppose that $[(I, m), (J, n)]$ is a closed interval in the dual poset P_N^*. Given a sequence of labels*

$$(l_1, \ldots, l_k)$$

there is at most one maximal chain σ in the closed interval such that

$$\lambda(\sigma) = (l_1, \ldots, l_k)\,.$$

When it exists, this chain must be equal to

$$(I, m) \lessdot (I \smallsetminus \{\ell_1\}, n_1) \lessdot \cdots \lessdot (I \smallsetminus \{\ell_1, \ldots, \ell_{k-1}\}, n_{k-1}) \lessdot (J, n)$$

where $\ell_i = |l_i|$, the set $I \smallsetminus J = \{\ell_1, \ldots, \ell_k\}$ and

$$n_i = \begin{cases} \mathbf{g}(x_{\ell_i} n_{i-1}) & \text{if } l_i > 0 \\ n_{i-1} & \text{if } l_i < 0 \end{cases}$$

for $1 \leq i \leq k$ with $n_0 = m$ and $n_k = n$.

Suppose that $(J, n) < (I, m)$ is a pair of comparable admissible symbols. Then $n = \mathbf{g}(x_{C'}m)$ for some $C' \subseteq I \smallsetminus J$. Let $C = \{c \in C' \mid c \leq \max(n)\}$. Then by associativity and [14, Lemma 1.2] we have $n = \mathbf{g}(x_{C'}m) = \mathbf{g}(x_{C' \smallsetminus C}\mathbf{g}(x_C m)) = \mathbf{g}(x_C m)$. In this way, any representation of $n = \mathbf{g}(x_{C'}m)$ may be reduced to $n = \mathbf{g}(x_C m)$ under the conditions above.

Notation 3.8. *Implicit in all subsequent arguments is the convention that a representation $n = \mathbf{g}(x_C m)$ is written in reduced form.*

Lemma 3.9. *For a reduced representation of $n = \mathbf{g}(x_C m)$ the set C is the unique subset of minimum cardinality among all $C' \subseteq I \smallsetminus J$ for which $n = \mathbf{g}(x_{C'} m)$.*

Proof. Suppose that C is not the subset of $I \smallsetminus J$ with smallest cardinality, namely that there exists $D \subseteq I \smallsetminus J$ with $|D| < |C|$ and $n = \mathbf{g}(x_D m)$. By definition, $x_C \cdot m = n \cdot y$ and $x_D \cdot m = n \cdot u$ where $\max(n) \leq \min(y)$ and $\max(n) \leq \min(u)$. The assumption of $|D| < |C|$ implies that there exists $c \in C$ such that $c \notin D$. Rearranging and combining the two equations above, we arrive at the equality $x_C \cdot u = x_D \cdot y$. This equality allows us to conclude that x_c divides y since it cannot divide x_D. By definition, $\max(n) \leq \min(y)$ and therefore $\max(n) \leq c$. Further, since we assumed that $n = \mathbf{g}(x_C m)$ possessed the property that $c \leq \max(n)$ we have $c \leq \max(n) \leq c$ so that $\max(n) = c$. This equality also has implications for x_D and u, namely that $c \leq \min(u)$ and $\max(D) \leq \max(n) = c$ so that $\max(D) < c$ since $c \notin D$. However, $c < \max(D) \leq \max(n) = c$ is a contradiction, and our original supposition that such a D exists is false. If C and D are distinct subsets of $I \smallsetminus J$ with $|C| = |D|$ and $n = \mathbf{g}(x_C m) = \mathbf{g}(x_D m)$ then there is a $c \in C$ and $d \in D$ for which $c \notin D$ and $d \notin C$. As before, we use the equality $x_C \cdot u = x_D \cdot y$ and now conclude that x_c divides y and x_d divides u. Therefore, $c \leq \max(C) \leq \max(n) \leq \min(y) \leq c$ and similarly $d \leq \max(D) \leq \max(n) \leq \min(u) \leq d$ so that $\max(n) = c = d$, ☐

Proof of Theorem 3.6.

To prove the dual EL-Shellability of P_N, recall that for the poset of admissible symbols P_N, we have comparability in the dual poset given by $(I, m) < (J, n) \in P_N^*$ if and only if $(J, n) < (I, m) \in P_N$. We proceed with the proof by considering the various types of closed intervals that appear in the dual poset P_N^*.

Case 1: Consider the closed interval $\big[(I, m), \hat{0}\big]$. Write $I = \{d_1, \ldots, d_t\}$ so that $d_j < d_{j+1}$, for every $j = 1, \ldots, t$. The maximal chain

$$\sigma = (I, m) \lessdot (I \smallsetminus \{d_t\}, m) \lessdot (I \smallsetminus \{d_t, d_{t-1}\}, m) \lessdot \cdots \lessdot (\varnothing, m) \lessdot \hat{0}$$

has the increasing label

$$(-d_t, -d_{t-1}, \ldots, -d_1, 0)\,.$$

Consider a maximal chain $\tau \in \big[(I,m),\hat{0}\big]$ where $\tau \neq \sigma$. If each label in the sequence $\lambda(\tau)$ (except the label of coverings of the form $(\varnothing, n) \lessdot \hat{0}$) is negative, then the sequence $\lambda(\tau)$ cannot be increasing, for it must be a permutation of the sequence $\lambda(\sigma)$ where the rightmost label 0 is fixed. If any label within the sequence $\lambda(\tau)$ is positive, then again $\lambda(\tau)$ cannot be increasing since every maximal chain contains the labeled subchain

$$(\varnothing, n) \overset{0}{\lessdot} \hat{0}$$

for every $(I,m) < (\varnothing, n)$. Therefore, σ is the unique rising chain in the interval $\big[(I,m),\hat{0}\big]$. Further, $\lambda(\sigma)$ is lexicographically first among all chains in $\big[(I,m),\hat{0}\big]$ since $-d_t < \cdots < -d_1 < 0$.

Case 2: Consider the closed interval $[(I,m),(J,n)]$ of P_N^* where $(J,n) \neq \hat{0}$ and $n = m$. Again write $I \smallsetminus J = \{d_1, \ldots, d_t\}$ such that $d_1 < \cdots < d_t$. Every maximal chain σ in $[(I,m),(J,m)]$ has a label of the form

$$\big(-d_{\rho(t)}, \ldots, -d_{\rho(1)}\big)$$

where $\rho \in \Sigma_t$ is a permutation of the set $\{1, \ldots, t\}$. Therefore, the label

$$(-d_t, \ldots, -d_1)$$

corresponding to the identity permutation is the unique increasing label in $[(I,m),(J,m)]$ and is lexicographically first among all such labels.

Case 3: Consider the closed interval $[(I,m),(J,n)]$ of P_N^* where $(J,n) \neq \hat{0}$ and $m \neq n$. By Lemma 3.9, $n = \mathbf{g}(x_C m)$ for a unique $C \subseteq I \smallsetminus J$ where $\max(C) \leq \max(n)$ and the set C is of minimum cardinality. Writing the set $C = \{c_1, \ldots, c_q\}$ and $(I \smallsetminus J) \smallsetminus C = \{\ell_1, \ldots, \ell_t\}$ where $\ell_1 < \cdots < \ell_t$ and $c_1 < \cdots < c_q$, it follows that the sequence of edge labels $(-\ell_t, \ldots, -\ell_1, c_1, \ldots, c_q)$ is the increasing label of a maximal chain σ in $[(I,m),(J,n)]$.

Turning to uniqueness, suppose that $\tau \neq \sigma$ is also a chain which has a rising edge label. Then

$$\lambda(\tau) = (-d_p, \ldots, -d_1, s_1, \ldots, s_j) \tag{3.10}$$

where

$$\{s_1, \ldots, s_j\} \cup \{d_1, \ldots, d_p\} = \{c_1, \ldots, c_q\} \cup \{\ell_1, \ldots, \ell_t\} = I \smallsetminus J,$$

and $-d_p < \ldots < -d_1 < 0 < s_1 < \ldots < s_j$. Since $\tau \neq \sigma$, then $\lambda(\tau) \neq \lambda(\sigma)$ and in particular, $\{d_1, \ldots, d_p\} \neq \{\ell_1, \ldots, \ell_t\}$.

If there exists $\ell \in \{\ell_1, \ldots, \ell_t\}$ with the property that $\ell \notin \{d_1, \ldots, d_p\}$, we must have $\ell \in \{s_1, \ldots, s_j\}$ so that $\ell = s_i$ for some $i < j$ and the label $\lambda(\sigma)$ has the form

$$(-d_p, \ldots, -d_1, s_1, \ldots, \ell, \ldots, s_j)\,. \tag{3.11}$$

By the definition of $\mathbf{g}$, we have the equalities $x_C \cdot m = n \cdot y$ and $x_S \cdot m = n \cdot u$, which may be combined and simplified to arrive at the equation $x_C \cdot u = x_S \cdot y$. The assumption that $\ell \in S$ and $\ell \notin C$ implies that x_ℓ divides u so that $\max(n) \leq \ell$. It therefore follows that $\max(n) \leq \ell < s_{i+1} < \cdots < s_j$ when $\ell \neq s_j$ so that $\max(I \smallsetminus \{s_1, \ldots, \ell\}) = s_j > \max(n)$, which is a contradiction the admissibility of the symbol $(I \smallsetminus \{d_1, \ldots, d_p, s_1, \ldots, \ell\}, n)$. If $\ell = s_j$ then $\max(n) \leq s_j$ and therefore $n = \mathbf{g}(x_{s_1} \cdots x_j m) = \mathbf{g}(x_{s_1} \cdots x_{j-1} m)$ which implies that the symbol $(I \smallsetminus \{s_1, \ldots, s_{j-1}\}, \mathbf{g}(x_{s_1} \cdots x_{j-1} m))$ which would neccesarily precede (J, n) in the chain is not admissible. If there exists $d_g \in \{d_1, \ldots, d_p\}$ with $d_g \notin \{\ell_1, \ldots, \ell_t\}$ then a similar argument again provides a contradiction to admissibility.

We now prove that $\lambda(\sigma)$ is lexicographically smallest among all chains. Aiming for a contradiction, suppose that the label $\lambda(\sigma)$ is not lexicographically smallest so that there exists a maximal dual chain τ with $\lambda(\tau) < \lambda(\sigma)$. Without loss of generality, we may assume that $\lambda(\sigma)$ and $\lambda(\tau)$ differ at their leftmost label $-c$, where $-c < -\ell_t$. Such a c must be an element of the set C since $-\ell_t < \cdots < -\ell_1$ is inherent in the structure of $\lambda(\sigma)$. By construction, $c \in C$ implies that $c \leq \max(n)$ and utilizing the equations $x_C \cdot m = n \cdot y$ and $x_S \cdot m = n \cdot u$, to produce $x_C \cdot u = x_S \cdot y$, it follows that x_c divides y and therefore $c \leq \max(n) \leq \min(y) \leq c$ so that $\max(n) = c$. This forces the element $c = c_q$ for otherwise, the chain σ would contain the subchain $(I \smallsetminus \{\ell_1, \ldots, \ell_t, c_1, \ldots, c\}, n) < (I \smallsetminus J, n)$ where $(I \smallsetminus \{\ell_1, \ldots, \ell_t, c_1, \ldots, c\}, n)$ is not an admissible symbol.

The desired contradiction will be obtained within an investigation of each of the three possibilities for the relationship between $\deg_{x_c}(n)$ and $\deg_{x_c}(m)$.

Suppose $\deg_{x_c}(n) > \deg_{x_c}(m)$ so that $\deg_{x_c}(n) = \deg_{x_c}(m) + 1$, based upon the structure of the set I and the definition of the decomposition map $\mathbf{g}$. In this case, the chain τ cannot end in (J, n) since $-c$, the leftmost label of τ, labels the subchain $(I, m) \lessdot (I \smallsetminus \{c\}, m)$ and the x_c degree of every monomial appearing in the chain τ may not increase.

If $\deg_{x_c}(n) < \deg_{x_c}(m)$ then the unique decomposition $x_C \cdot m = n \cdot u$ implies that x_c divides u , for otherwise $c \in C$ implies that $\deg_{x_c}(n) = \deg_{x_c}(m) + 1$, a contradiction. The conclusion that x_c divides u allows $x_C \cdot m =$

$n \cdot u$ to be simplified to $x_{C'} \cdot m = n \cdot u'$ where $C' = C \setminus \{c\}$ and $u' = u/x_c$. This contradicts the condition that C is the set of smallest cardinality for which $\mathbf{g}(x_C m) = n$.

Lastly, if $\deg_{x_c}(n) = \deg_{x_c}(m)$ we turn to the chain σ, whose rightmost label is c. The subchain with this label is $(I \setminus \{\ell_1, \ldots, \ell_t, c_1, \ldots, c_{j-1}\}, n') < (I \setminus J, n)$ where $x_c \cdot n' = n \cdot y$ where n does not contain this new factor of x_c. The monomial x_c therefore divides y and we can reduce $x_c \cdot n' = n \cdot y$ to $n' = n \cdot u'$ where $u' = u/x_c$, a contradiction to $n' \in G(N)$. This completes the proof. $\square$

With Theorem 3.6 established, we immediately have the following corollary.

Corollary 3.12. *Every interval of P_N which is of the form $\left[\hat{0}, (I, m)\right]$ is finite, dual EL-shellable and therefore shellable.*

4 The topology of P_N and properties of $\mathcal{D}(P_N)$

To establish the connection between the poset P_N and the sequence $\mathcal{D}(P_N)$, we recall the definition of *CW poset*, due to Björner [5].

Definition 4.1. *[5] A poset P is called a* CW poset *if*

1. *P has a least element $\hat{0}$,*
2. *P is nontrivial (has more than one element),*
3. *For all $x \in P \setminus \{\hat{0}\}$, the open interval $(\hat{0}, x)$ is homeomorphic to a sphere.*

After establishing this definition, Björner describes sufficient conditions for a poset to be a CW poset.

Proposition 4.2. *[5, Proposition 2.2] Suppose that P is a nontrivial poset such that*

1. *P has a least element $\hat{0}$,*
2. *every interval $[x, y]$ of length two has cardinality four,*
3. *For every $x \in P$ the interval $[\hat{0}, x]$ is finite and shellable.*

Then P is a CW poset.

With this proposition in hand, we now may conclude the following about the structure of P_N, the poset of admissible symbols.

Theorem 4.3. *The poset of admissible symbols P_N is a CW poset.*

Proof. The poset P_N has a least element by construction and each of its intervals $\left[\hat{0},(I,m)\right]$ is finite and shellable by Corollary 3.12. Thus, it remains to show that every closed interval in P_N of length two has cardinality four.

Case 1: Let $(J,n)=\hat{0}$ so that the set I is a singleton. It follows that the only poset elements in the open interior of the interval are $(\varnothing,m)$ and $(\varnothing,\mathbf{g}(x_I m))$.

Case 2: Let $(J,n)\neq\hat{0}$ and suppose that $[(J,n),(I,m)]$ is a closed interval of length two in the poset of admissible symbols, P_N. Since the interval is of length two, the set J has the form $I\setminus\{i_0,i_1\}$ for some $i_0<i_1\in I$. Further, any poset element in the interval must have either $I\setminus\{i_0\}$ or $I\setminus\{i_1\}$ as its first coordinate, for these sets are the only subsets of I which contain $I\setminus\{i_0,i_1\}$.

Write $m=m'x_{i_2}x_{i_3}$ where $\max(m')\leq i_2\leq i_3$. We must now consider each of the possible orderings for the elements of the (multi) set $\{i_0,i_1,i_2,i_3\}$ to ascertain the choices available for the monomial n. Our assumptions of the inequalities $i_0<i_1$ and $i_2\leq i_3$ together with the admissibility of the symbol (I,m) imply that $i_1\leq\max(I)<\max(m)\leq i_3$. Hence, determining the number of orderings amounts to producing a count of the number of orderings for elements of the set $\{i_0,i_1,i_2\}$, of which there are three, since $i_0<i_1$.

Subcase 2.1: Suppose that $i_0<i_1<i_2\leq i_3$.

If $n=m$, then the poset elements which are contained in the open interior of the interval are forced to be $(I\setminus\{i_0\},m)$ and $(I\setminus\{i_1\},m)$.

If $n=\mathbf{g}(x_{i_0}m)$ and $\max(I\setminus\{i_0\})<\max(\mathbf{g}(x_{i_0}m))$ then the symbol $(I\setminus\{i_0\},\mathbf{g}(x_{i_0}m))$ is admissible, so that it is in the open interior of the interval along with the admissible symbol $(I\setminus\{i_1\},m)$. The symbol $(I\setminus\{i_0\},m)$ is not comparable to $(I\setminus\{i_0\},\mathbf{g}(x_{i_0}m))$ due to the absence of the value i_0. The symbol $(I\setminus\{i_1\},\mathbf{g}(x_{i_1}m))$ is also not comparable to $(I\setminus\{i_0\},\mathbf{g}(x_{i_0}m))$, for if it were then either $\mathbf{g}(x_{i_0m})=\mathbf{g}(x_{\varnothing}\mathbf{g}(x_{i_1}m))=\mathbf{g}(x_{i_1}m)$ or $\mathbf{g}(x_{i_0}x_{i_1}m)=n=\mathbf{g}(x_{i_0}m)$. The first equality is impossible since Lemma 3.9 guarantees that $\{i_0\}$ is the unique set containing one element for which $n=\mathbf{g}(x_{i_0}m)$. The second equality also can not occur since Lemma 1.2 of [14] guarantees

monomial equality $\mathbf{g}(x_{i_0}x_{i_1}m) = \mathbf{g}(x_{i_0}m)$ if and only if $\max(n) \leq \min(x_{i_1}) = i_1$, which would contradict the assumption that $(I \smallsetminus \{i_0\}, n)$ is an admissible symbol.

If $n = \mathbf{g}(x_{i_0}m)$ and $\max(I \smallsetminus \{i_0\}) \geq \max(\mathbf{g}(x_{i_0}m))$ then the symbol $(I \smallsetminus \{i_0\}, \mathbf{g}(x_{i_0}m))$ is not admissible and is not an element of P_N. However, we are assuming that the symbol $(I \smallsetminus \{i_0, i_1\}, \mathbf{g}(x_{i_0}m))$ is admissible, so that $\max(\mathbf{g}(x_{i_0}m)) \leq i_1$ and via Lemma 1.2 of [14], we have the monomial equality $\mathbf{g}(x_{i_0}\mathbf{g}(x_{i_1}m)) = \mathbf{g}(x_{i_1}m)$. Therefore, $(I \smallsetminus \{i_0\}, \mathbf{g}(x_{i_0}m)) = (I \smallsetminus \{i_0\}, \mathbf{g}(x_{i_0}x_{i_1}m))$ and the symbols $(I \smallsetminus \{i_1\}, m)$ and $(I \smallsetminus \{i_1\}, \mathbf{g}(x_{i_1}m))$ are each contained in the interval. Since $n = \mathbf{g}(x_{i_0}m)$, the symbol $(I \smallsetminus \{i_0\}, m)$ is not comparable to $(I \smallsetminus \{i_0\}, \mathbf{g}(x_{i_0}m))$.

If $n = \mathbf{g}(x_{i_1}m)$ then the symbol $(I \smallsetminus \{i_0\}, m)$ is certainly contained in the closed interval. Further, $(I \smallsetminus \{i_1\}, \mathbf{g}(x_{i_1}m))$ must be admissible for were it not, then the assumption of admissibility for $(I \smallsetminus \{i_0, i_1\}, \mathbf{g}(x_{i_1}m))$ implies that $i_0 \geq \max(\mathbf{g}(x_{i_1}m)) \geq \min(\mathbf{g}(x_{i_1}m)) \geq i_1$, a contradiction to the initial stipulation that $i_0 < i_1$. The symbol $(I \smallsetminus \{i_1\}, m)$ is incomparable to $(I \smallsetminus \{i_0, i_1\}, \mathbf{g}(x_{i_1}m))$ and were $(I \smallsetminus \{i_0\}, \mathbf{g}(x_{i_0}m))$ comparable to $(I \smallsetminus \{i_0, i_1\}, \mathbf{g}(x_{i_1}m))$, then either $\mathbf{g}(x_{i_1 m}) = \mathbf{g}(x_{\varnothing}\mathbf{g}(x_{i_0}m)) = \mathbf{g}(x_{i_0 m})$ or $\mathbf{g}(x_{i_0}x_{i_1}m) = n = \mathbf{g}(x_{i_0}m)$. The first equality contradicts Lemma 3.9 and the second may be used to arrive at a contradiction to the admissibility of $(I \smallsetminus \{i_1\}, \mathbf{g}(x_{i_1}m))$. These arguments are similar to those used in when $n = \mathbf{g}(x_{i_0}m)$ and $\max(I \smallsetminus \{i_0\}) < \max(\mathbf{g}(x_{i_0}m))$.

If $n = \mathbf{g}(x_{i_0}x_{i_1}m)$ and $n = \mathbf{g}(x_{i_0}x_{i_1}m) \neq \mathbf{g}(x_{i_0}m)$ then the symbols $(I \smallsetminus \{i_0\}, \mathbf{g}(x_{i_0}m))$ and $(I \smallsetminus \{i_1\}, \mathbf{g}(x_{i_1}m))$ are admissible and are contained in the open interior of the interval. Clearly, the symbols $(I \smallsetminus \{i_0\}, m)$ and $(I \smallsetminus \{i_1\}, m)$ are not comparable to $(I \smallsetminus \{i_0, i_1\}, \mathbf{g}(x_{i_0}x_{i_1}m))$ in this instance. If $n = \mathbf{g}(x_{i_0}x_{i_1}m)$ and $n = \mathbf{g}(x_{i_0}x_{i_1}m) = \mathbf{g}(x_{i_0}m)$, then we are reduced to an already resolved case.

For each of these four choices of n, the interval has four elements.

Subcase 2.2 We now consider the two remaining orderings $i_0 < i_2 \leq i_1 < i_3$ and $i_2 \leq i_0 < i_1 < i_3$. Under each of these orderings, we have $\deg_{x_{i_3}}(m) = 1$ and in light of Lemma 1.3 of [14] if $n \neq m$ we have $\max(n) < \max(m) = i_3$ and in turn that $\max(n) \leq i_1$.

If $n = m$, then the poset elements which are contained in the open interior of the interval are forced to be $(I \smallsetminus \{i_0\}, m)$ and $(I \smallsetminus \{i_1\}, m)$.

If $n = \mathbf{g}(x_{i_0}m)$ then $\max(n) \leq i_1$ implies that the symbol $(I \smallsetminus \{i_0\}, \mathbf{g}(x_{i_0}m))$ is not admissible and is not an element of P_N. However, we are assuming that

the symbol $(I \smallsetminus \{i_0, i_1\}, \mathbf{g}(x_{i_0}m))$ is admissible, so that $\max(\mathbf{g}(x_{i_0}m)) \leq i_1$ and again using Lemma 1.2 of [14], we have $\mathbf{g}(x_{i_0}\mathbf{g}(x_{i_1}m)) = \mathbf{g}(x_{i_0}m)$. Therefore, $(I\smallsetminus\{i_0\}, \mathbf{g}(x_{i_0}m)) = (I\smallsetminus\{i_0\}, \mathbf{g}(x_{i_0}x_{i_1}m))$ and the symbols $(I\smallsetminus\{i_1\}, m)$ and $(I\smallsetminus\{i_1\}, \mathbf{g}(x_{i_1}m))$ are each contained in the interval. Since $n = \mathbf{g}(x_{i_0}m)$, the symbol $(I \smallsetminus \{i_0\}, m)$ is not comparable to $(I \smallsetminus \{i_0\}, \mathbf{g}(x_{i_0}m))$.

If $n = \mathbf{g}(x_{i_1}m)$ then the symbol $(I \smallsetminus \{i_0\}, m)$ is certainly contained in the closed interval. Further, $(I \smallsetminus \{i_1\}, \mathbf{g}(x_{i_1}m))$ must be admissible for were it not, then the assumption of admissibility for $(I \smallsetminus \{i_0, i_1\}, \mathbf{g}(x_{i_1}m))$ implies that $i_0 \geq \max(\mathbf{g}(x_{i_1}m)) \geq \min((\mathbf{g}(x_{i_1}m))) \geq i_1$, a contradiction to the initial stipulation that $i_0 < i_1$. The symbol $(I \smallsetminus \{i_1\}, m)$ is incomparable to $(I \smallsetminus \{i_0, i_1\}, \mathbf{g}(x_{i_1}m))$ and were $(I \smallsetminus \{i_0\}, \mathbf{g}(x_{i_0}m))$ comparable to $(I \smallsetminus \{i_0, i_1\}, \mathbf{g}(x_{i_1}m))$, then either $\mathbf{g}(x_{i_1m}) = \mathbf{g}(x_{\varnothing}\mathbf{g}(x_{i_0}m)) = \mathbf{g}(x_{i_0m})$ or $\mathbf{g}(x_{i_0}x_{i_1}m) = n = \mathbf{g}(x_{i_0}m)$. The first equality contradicts Lemma 3.9 and the second may be used to arrive at a contradiction to the admissibility of $(I \smallsetminus \{i_1\}, \mathbf{g}(x_{i_1}m))$. These arguments are similar to those used in the case when $n = \mathbf{g}(x_{i_0}m)$ and $\max(I \smallsetminus \{i_0\}) < \max(\mathbf{g}(x_{i_0}m))$.

Again, for each of these three choices of n, the interval has four elements.

□

We now analyze the vector spaces which are present in the sequence $\mathcal{D}(P_N)$ at the level of individual poset elements. In order to do so, we recall the following combinatorial results. As is standard, we write $\bar{P} = P \smallsetminus \{\hat{0}, \hat{1}\}$.

Theorem 4.4 ([4, 8])**.** *If a bounded poset P is EL-shellable, then the lexicographic order of the maximal chains of P is a shelling of $\Delta(P)$. Moreover, the corresponding order of the maximal chains of $\bar{P}$ is a shelling of $\Delta(\bar{P})$.*

Theorem 4.5 ([8])**.** *Suppose that P is a poset for which $\hat{P} = P \cup \{\hat{0}, \hat{1}\}$ admits an EL-labeling. Then P has the homotopy type of a wedge of spheres. Furthermore, for any fixed EL-labeling:*

i. $\widetilde{H}_i(\Delta(P), \mathbb{Z}) \cong \mathbb{Z}^{\#\text{falling chains of length } i+2}$,

ii. bases for i-dimensional homology (and cohomology) are induced by the falling chains of length $i+2$.

In the analysis that follows, we again examine the dual poset P_N^* and focus our attention on the collection of closed intervals of the form $[(I, m), \hat{0}]$, to each of which we apply Theorem 4.5. Indeed, for each admissible symbol $(I, m) \in P_N^*$ where $|I| = q$, the open interval $((I, m), \hat{0})$ is homeomorphic to

a sphere of dimension $q-1$ since P_N is a CW poset. Further, the EL-labeling of $[(I,m),\hat{0}]$ guarantees that the unique generator of $\widetilde{H}_{q-1}(\Delta_{I,m},\Bbbk)$ is induced by a unique falling chain of length $q+1$. In the discussion that follows, we use the EL-shelling of Definition 3.4 to produce a canonical generator of $\widetilde{H}_{q-1}(\Delta_{I,m},\Bbbk)$ as a linear combination in which each facet of $\Delta_{I,m}$ occurs with coefficient $+1$ or -1.

To begin, consider a maximal chain $(I,m) \lessdot \sigma \lessdot \hat{0}$ which is of length $q+1$ and appears in the dual closed interval $\left[(I,m),\hat{0}\right]$ and write the label of this chain as

$$\left(l_1^\sigma,\ldots,l_q^\sigma,0\right). \tag{4.6}$$

We note that $I=\{|l_1^\sigma|,\ldots,|l_q^\sigma|\}$ and write

$$\varepsilon_\sigma \;=\; \operatorname{sgn}(\rho_\sigma)\cdot\operatorname{sgn}\left(\prod_{t=1}^{q} l_q\right) \tag{4.7}$$

where $\rho_\sigma \in \Sigma_q$ is the permutation arranging the sequence

$$|l_1^\sigma|,\ldots,|l_q^\sigma|$$

in *increasing* order. We endow the corresponding chain σ in $\left((I,m),\hat{0}\right)$ with this sign ε_σ and refer to it as the *sign* of σ.

The unique maximal chain τ in $\left[(I,m),\hat{0}\right]$ which has a decreasing label is the chain consisting of admissible symbols having at each stage a different monomial as their second coordinate and the sequence of sets

$$I, I\smallsetminus\{i_q\}, I\smallsetminus\{i_{q-1},i_q\},\ldots,\{i_1,i_2\},\{i_1\},\varnothing$$

as their first coordinate. The unique falling chain $\tau\in\left[(I,m),\hat{0}\right]$ is therefore

$$(I,m)\lessdot(I_q,m_q)\lessdot(I_{q-1,q},m_{q-1,q})\lessdot\cdots\lessdot(I_{2,\ldots,q},m_{2,\ldots,q})\lessdot(\varnothing,m_{1,\ldots,q})\lessdot\hat{0}$$

where $I=\{i_1,\ldots,i_q\}$ with $i_1<\ldots<i_q$ and for $j=1,\ldots,q$, the set $I_{j,\ldots,q}=I\smallsetminus\{i_j,\ldots,i_q\}$ and the monomial $m_{j,\ldots,q}=\mathbf{g}(x_{i_j}\cdots x_{i_q}m)$. The label of the chain τ is therefore

$$(i_q,\ldots,i_1,0)$$

and is decreasing. If there were another such chain with decreasing label, then such a chain would be counted by Theorem 4.5 and $\left(\hat{0},(I,m)\right)$ would

not have the homotopy type of a sphere, a contradiction to the fact that P_N is a CW poset. In the context of the shelling order produced by the EL-shelling above, the chain τ appears lexicographically last among all maximal chains in the dual interval and is therefore the unique homology facet of $\Delta_{I,m}$.

Definition 4.8. *For an admissible symbol* (I, m)*, set*

$$f(I,m) = \sum_{\sigma \in \left((I,m),\hat{0}\right)} \varepsilon_\sigma \cdot \sigma,$$

the linear combination of all maximal chains of the open interval $\big((I,m),\hat{0}\big)$ *with coefficients given by (4.7).*

Viewing the maximal chains of $\big((I,m),\hat{0}\big)$ as facets in the order complex $\Delta_{I,m}$ we now establish the following.

Lemma 4.9. *The sum* $f(I,m)$ *is a* $q-1$*-dimensional cycle in* $\widetilde{H}_{q-1}(\Delta_{I,m}, \mathbb{k})$ *which is not the boundary of any* q*-dimensional face.*

Proof. The maximal chains in the open interval $\big((I,m),\hat{0}\big)$ are each of length $q-1$, so that no q-dimensional faces are present in $\Delta_{I,m}$. Thus, $f(I,m)$ cannot be the boundary of a q-dimensional face of $\Delta_{I,m}$.

We now show that $f(I,m)$ is a $q-1$-dimensional cycle. Suppose that σ is a maximal chain in $\big((I,m),\hat{0}\big)$ and let (J,n) be an element of said chain. We exhibit a unique chain σ' which also appears in $f(I,m)$ and differs from σ only at the element (J,n).

Indeed, consider the chain $(I,m) \lessdot \sigma \lessdot \hat{0}$ along with its subchain $(J_1,n_1) \lessdot (J,n) \lessdot (J_2,n_2)$. In the proof of Theorem 4.3, each closed interval of length two was shown to be of cardinality four, and therefore there exists a unique $(J',n') \in [(J_1,n_1),(J_2,n_2)]$ which is not equal to (J,n). Defining σ' by removing (J,n) and replacing it with (J',n'), we have constructed the desired chain.

We claim that for the chains σ and σ', the associated signs ε_σ and $\varepsilon_{\sigma'}$ are opposite to one another.

If $(J_2,n_2) = \hat{0}$ then $(J_1,n_1) = (\{j\},n_1)$ for some j. Thus, $(J,n) = (\varnothing,n)$ and $(J',n') = (\varnothing,n')$ so that the chains σ and σ' have the same corresponding permutation ρ. Since either $n_1 = n$ or $n_1 = n'$, without loss of generality we assume that $n_1 = n$ so that $n' = \mathbf{g}(x_j n)$. Therefore, the subchain $(\{j\},n) \lessdot (\varnothing,n) \lessdot \hat{0}$ has $-j$ as its label, while $(\{j\},n) \lessdot (\varnothing,n') \lessdot \hat{0}$

has j as its label. This is the only difference in the labels $\lambda(\sigma)$ and $\lambda(\sigma')$ and $\varepsilon_\sigma \neq \varepsilon_{\sigma'}$ is forced.

If $(J_2, n_2) \neq \hat{0}$ then for each case that appears in the classification of intervals of length two described in the proof of Theorem 4.3, we can compute $\varepsilon_\sigma \neq \varepsilon_{\sigma'}$.

When the differential d in the reduced chain complex $\widetilde{\mathcal{C}}_\bullet(\Delta_{I,m})$ is applied to the sum $f(I,m)$, each term appears twice with opposite signs, so that $d(f(I,m)) = 0$ making $f(I,m)$ a $q-1$-dimensional cycle in $\widetilde{H}_{q-1}(\Delta_{I,m}, \Bbbk)$ as claimed. ☐

5 Proof of Theorem 2.5

With the choice for the bases of the vector spaces in $\mathcal{D}(P_N)$ established, we now turn to the proof that the poset P_N supports the minimal free resolution of R/N. We first analyze the action of the differential of $\mathcal{D}(P_N)$ when it is applied to an arbitrary basis element $f(I,m)$.

Lemma 5.1. *The map $\varphi_{q+1}^{(I,m),(J,n)}$ sends a basic cycle $f(I,m)$ to the element $(-1)^{p+\delta_{m,n}} \cdot f(J,n)$, where $I = \{i_1, \ldots, i_q\}$, the relationship $I \setminus J = \{i_p\}$ holds and*

$$\delta_{m,n} = \begin{cases} 1 & \text{if } m \neq n \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Write d for the simplicial differential in the reduced chain complex $\widetilde{\mathcal{C}}_\bullet(\Delta_{I,m})$. The open interval $\big((I,m), \hat{0}\big)$ may be realized as the union of half-closed intervals $\big[(J,n), \hat{0}\big)$, so that the order complex of each half-closed interval is a cone with apex (J,n). Applying the differential to the sum of all facets contained in the interval produces the boundary of the cone, which in this case is the order complex of $\big((J,n), \hat{0}\big)$. Indeed, when d is applied to the sum

$$\upsilon = \sum_{\sigma \in [(J,n), \hat{0})} \varepsilon_\sigma \cdot \sigma,$$

the faces in which the element (J,n) remains appear twice and have opposite signs as described in the proof of Lemma 4.9. Thus, the only faces that remain in the expansion of $d(\upsilon)$ are of the form $\bar{\sigma} = \sigma \setminus \{(J,n)\}$.

Precisely,

$$\begin{aligned}
\varphi_{q+1}^{(I,m),(J,n)}(f(I,m)) &= \left[d\left(\sum_{\sigma\in\left[(J,n),\hat{0}\right)} \varepsilon_\sigma\cdot\sigma \right)\right] \qquad (5.2)\\
&= \left[\sum_{\sigma\in\left[(J,n),\hat{0}\right)} \varepsilon_\sigma\cdot\bar{\sigma} \right]\\
&= \left[\sum_{\bar{\sigma}\in\left((J,n),\hat{0}\right)} \varepsilon_\sigma\cdot\bar{\sigma} \right].
\end{aligned}$$

The facet $\bar{\sigma}$ has an associated permutation $\rho_{\bar{\sigma}}\in\Sigma_{q-1}$, and using elementary properties of permutation signs, we have $\operatorname{sgn}(\rho_\sigma)=(-1)^{p+1}\cdot\operatorname{sgn}(\rho_{\bar{\sigma}})$, where $I\setminus J=\{i_p\}$. Considering the definition of ε_σ, for each (J,n) for which $(I,m)\lessdot(J,n)\in P_N^*$ we now have

$$\begin{aligned}
\varepsilon_\sigma &= \operatorname{sgn}(\rho_\sigma)\cdot\operatorname{sgn}\left(\prod_{t=1}^{q} l_q\right)\\
&= (-1)^{p+1}\cdot\operatorname{sgn}(\rho_{\bar{\sigma}})\cdot\operatorname{sgn}\left(\prod_{t=2}^{q} l_q\right)\cdot\operatorname{sgn}(l_1)\\
&= (-1)^{p+1}\cdot\operatorname{sgn}(l_1)\cdot\varepsilon_{\bar{\sigma}}\\
&= (-1)^{p+\delta_{m,n}}\cdot\varepsilon_{\bar{\sigma}}
\end{aligned}$$

since $\operatorname{sgn}(l_1)=1$ if $n\neq m$ and $\operatorname{sgn}(l_1)=-1$ if $n=m$.

Therefore, Equation (5.2) becomes

$$
\begin{aligned}
\varphi_{q+1}^{(I,m),(J,n)}(f(I,m)) &= \left[\sum_{\bar{\sigma}\in\left((J,n),\hat{0}\right)} \varepsilon_{\sigma}\cdot\bar{\sigma}\right] \\
&= \left[\sum_{\bar{\sigma}\in\left((J,n),\hat{0}\right)} (-1)^{p+\delta_{m,n}}\cdot\varepsilon_{\bar{\sigma}}\cdot\bar{\sigma}\right] \\
&= (-1)^{p+\delta_{m,n}}\cdot\left[\sum_{\bar{\sigma}\in\left((J,n),\hat{0}\right)} \varepsilon_{\bar{\sigma}}\cdot\bar{\sigma}\right] \\
&= (-1)^{p+\delta_{m,n}}\cdot f(J,n)
\end{aligned}
$$

which proves the lemma. $\square$

As described in Section 2, the map φ_{q+1} is defined componentwise on the one-dimensional $\Bbbk$-vectorspace $\mathcal{D}_{q+1,(I,m)}$ for each poset element (I,m). Using the conclusion of Lemma 5.1, we immediately have

$$
\varphi_{q+1}|_{\mathcal{D}_{q+1,(I,m)}} = \varphi_{q+1,(I,m)}(f(I,m)) = \sum_{(J,n)\lessdot(I,m)} (-1)^{p+\delta_{m,n}} f(J,n) \tag{5.3}
$$

where $I=\{i_1,\dots,i_q\}$ and $i_1<\dots<i_q$ and $J=I\smallsetminus\{i_p\}$.

Recall that the poset map $\eta : P_N \longrightarrow \mathbb{N}^n$ is defined as $(I,m)\mapsto \mathrm{mdeg}(x_I m)$, so that we can homogenize the sequence of vector spaces $\mathcal{D}(P_N)$ to produce

$$
\mathcal{F}(\eta) : 0 \longrightarrow F_d \xrightarrow{\partial_d^{\mathcal{F}(\eta)}} F_{d-1} \longrightarrow \dots \longrightarrow F_1 \xrightarrow{\partial_1^{\mathcal{F}(\eta)}} F_0,
$$

a sequence of multigraded modules. More precisely, for $q\geq 0$ and a poset element $(I,m)\neq\hat{0}$ where $I=\{i_1,\dots,i_q\}$ and $i_1<\dots<i_q$, the differential $\partial^{\mathcal{F}(\eta)}$ acts on a basis element $f(I,m)$ of the free module F_{q+1} via the formula

$$
\begin{aligned}
\partial_{q+1}^{\mathcal{F}(\eta)}(f(I,m)) = &\sum_{(J,n')\lessdot(I,m)} (-1)^{p+\delta_{m,n'}} x^{\eta(I,m)-\eta(J,n')}\cdot f(J,n') \\
= &\sum_{(J,m)\lessdot(I,m)} (-1)^{p} x_{i_p}\cdot f(J,m) - \sum_{(J,n)\lessdot(I,m)} (-1)^{p}\frac{x_{i_p}m}{\mathbf{g}(x_{i_p}m)}\cdot f(J,n)
\end{aligned} \tag{5.4}
$$

where p takes the same value as in 5.3, so that $I \setminus \{i_p\} = J$.

It remains to show that $\mathcal{F}(\eta)$ is a minimal exact complex, and to do so we identify it as the Eliahou-Kervaire resolution.

Definition 5.5. *The* Eliahou-Kervaire minimal free resolution *[14] of a stable ideal N is*

$$\mathcal{E} : 0 \longrightarrow E_d \xrightarrow{\partial_d^{\mathcal{E}}} E_{d-1} \longrightarrow \cdots \longrightarrow E_1 \xrightarrow{\partial_1^{\mathcal{E}}} E_0$$

where $E_0 = R$ is the free module of rank one with basis 1 and for $q \geq 0$, E_{q+1} has as basis the admissible symbols

$$\{e(I,m) : I = \{i_1, \ldots, i_q\}, \max(I) < \max(m)\}.$$

When applied to a basis element, the differential of $\mathcal{E}$ takes the form

$$\begin{aligned} \partial_{q+1}^{\mathcal{E}}\left(e(I,m)\right) &= \sum_{p=1}^{q}(-1)^p x_{i_p} \cdot e\left(I \setminus \{i_p\}, m\right) \\ &\quad - \sum_{p=1}^{q}(-1)^p \frac{x_{i_p} m}{\mathbf{g}(x_{i_p} m)} \cdot e\left(I \setminus \{i_p\}, \mathbf{g}(x_{i_p} m)\right) \end{aligned}$$

Where we define $e(I \setminus \{i_p\}, \mathbf{g}(x_p m)) = 0$ when $\max(I \setminus \{i_p\}) \geq \max(\mathbf{g}(x_p m))$ (i.e. the symbol is inadmissible).

We now are in a position to prove the main result of this paper.

Proof of Theorem 2.5. The Eliahou-Kervaire admissible symbols index the multigraded free modules in the complexes $\mathcal{E}$ and $\mathcal{F}(\eta)$ and therefore the generators of these modules are in one to one correspondence with one another. Further, comparing Definition 5.5 and Equation 5.4, $\partial^{\mathcal{E}}$ and $\partial^{\mathcal{F}(\eta)}$ have identical behavior on basis elements. The minimality and exactness of $\mathcal{E}$ implies the minimality and exactness of $\mathcal{F}(\eta)$ so that $\mathcal{F}(\eta)$ is a minimal poset resoution of R/N. ☐

6 A Minimal Cellular Resolution of R/N

In this section we exhibit a minimal cellular resolution for an arbitrary stable monomial ideal. Cellular resolutions of monomial ideals have received

considerable attention in the literature and recently Mermin [16] has shown that the Eliahou-Kervaire resolution $\mathcal{E}$ is cellular using methods distinct from those depicted here. The techniques of Discrete Morse Theory have also been used by Batzies and Welker [1] to produce a minimal cellular resolution of stable modules, which contain the class of stable ideals.

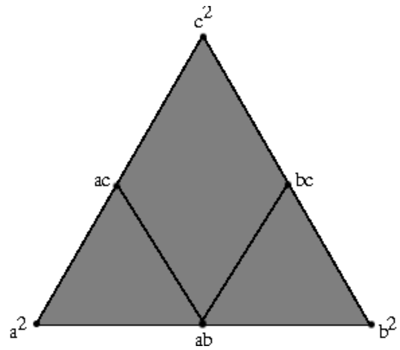
More generally, methods for determining whether a given monomial ideal admits a minimal cellular (or CW) resolution remain an open question, although Velasco [20] has shown that there exist monomial ideals whose minimal free resolutions are not even supported on a CW complex.

The technique described here is an example of a more general approach which interprets CW resolutions of monomial ideals through the theory of poset resolutions. This approach is described in [10], and is distinct from both the method of [1] concerning stable modules and the method of [16] which is specific to stable ideals. We begin by recalling a fundamental result due to Björner.

Proposition 6.1. *[5, Proposition 3.1] A poset P is a CW poset if and only if it is isomorphic to the face poset of a regular CW complex.*

In the case of the poset of admissible symbols P_N, we interpret Björner's proof explicitly to produce the corresponding regular CW complex X_N. On the level of cells, $\hat{0} \in P_N$ corresponds to the empty cell and each admissible symbol (I, m) of P_N corresponds to a closed cell $X_{I,m}$ of dimension $|I|$ for which $P(X_{I,m}) = [\hat{0}, (I, m)]$. Taking $X_N = \bigcup X_{I,m}$ we have an isomorphism of posets $P(X_N) \cong P_N$. The regular CW complex X_N also comes equipped with a $\mathbb{Z}^n$ grading by realizing the map $\eta : P_N \longrightarrow \mathbb{N}^n$ of Theorem 2.5 as a map $\eta : X_N \longrightarrow \mathbb{N}^n$ where a cell $X_{I,m} \mapsto \eta(I, m) = \mathrm{mdeg}(x_I m)$.

Example 6.2. *The stable ideal $N = \langle a, b, c\rangle^2 = \langle a^2, ab, ac, b^2, bc, c^2\rangle$ has minimal resolution supported by X_N, the regular CW complex depicted below, which has six 0-cells, eight 1-cells and three 2-cells. The face poset of this cell complex $P(X_N)$ is isomorphic to the poset of admissible symbols P_N given in Example 3.5.*



We recall the following well-known definition to which we incorporate the information given by the poset map η. For a more comprehensive view of cellular and CW resolutions, see [1, 3, 20].

Definition 6.3. *A complex of multigraded R-modules, $\mathcal{F}_N$, is said to be a* cellular resolution of R/N *if there exists an $\mathbb{N}^n$-graded regular CW complex X such that:*

1. *For all $i \geq 0$, the free module $(\mathcal{F}_N)_i$ has as its basis the $i-1$ dimensional cells of X.*
2. *For a basis element $e \in (\mathcal{F}_N)_i$, one has $\mathrm{mdeg}(e) = \eta(e)$,*
3. *The differential ∂ of $\mathcal{F}_N$ acts on a basis element $e \in (\mathcal{F}_N)_i$ as*

$$\partial(e) = \sum_{\substack{e' \subset e \subset X \\ \dim(e) = \dim(e')+1}} c_{e,e'} \cdot x^{\eta(e)-\eta(e')} \cdot e'$$

   where $c_{e,e'}$ is the coefficient of the cell e' in the differential of e in the cellular chain complex of X.

With this definition in hand, we are now able to reinterpret Theorem 2.5 in our final result.

Theorem 6.4. *Suppose that N is a stable monomial ideal. Then the minimal free resolution $\mathcal{F}(\eta)$ is a minimal cellular resolution of R/N.*

Proof. Conditions 1 and 2 of Definition 6.3 are clear from the structure of X_N, its correspondence to the poset P_N and the construction of the resolution $\mathcal{F}(\eta)$. It therefore remains to verify that condition 3 is satisfied. The main result in [10] provides a canonical isomorphism between the complex $\mathcal{D}(P_N)$ and $\mathcal{C}(X_N)$, the cellular chain complex of X_N. Therefore, the differential of $\mathcal{F}(\eta)$ satisfies condition 3. □

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